Formalization in Constructive Type Theory of the Standardization Theorem for the Lambda Calculus using Multiple Substitution

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Outline

1. Introduction
2. Preliminaries
3. Proof of the Standardization Theorem
4. Proof of the Leftmost Reduction Theorem
Previous work: Formal metatheory of the Lambda Calculus using Stoughton’s substitution

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- Formalization of the Lambda Calculus in Agda using one sort of names for both free and bound variables.
- Multiple substitution based on Stoughton’s paper (1988).
- Structural inductive proofs for the Church-Rosser theorem and Subject Reduction.
- Library with definitions and lemmas for manipulating substitution. Fully checked in Agda.
Present work

Our goals

- Extend these metatheoretical results by proving:
  - Standardization Theorem for $\beta$-reduction
  - Leftmost Reduction Theorem

- Assess the extent at which the library can be reused for this development.

- Attempt to use structural induction only.
The Standardization Theorem

**Definition (Standard reduction sequence)**
A reduction sequence is said to be standard if successive redexes are contracted from left to right, possibly with some jumps.

**Theorem (Standardization)**
*If a term $M$ $\beta$-reduces to a term $N$, then there exists a standard $\beta$-reduction sequence from $M$ to $N$.*

**Corollary (Leftmost reduction)**
*If a term has a $\beta$ normal form, then the leftmost-outermost reduction strategy will find this normal form.*
Proofs of the Standardization Theorem

- **Barendregt 1982**
  - Uses residuals to define standard reductions.
  - Distinguishes between internal and head reductions.
  - Based on the FD and FD!

- **Takahashi 1995**
  - Follows a similar structure to Barendregt’s.
  - Relies on Martin-Löf’s parallel reductions to represent the reduction of a set of redexes.
  - Inductive structure.
• Inductive definition of $\beta$-reducibility with a standard sequence.
• Uses neither residuals nor the separation between internal and head reductions.
• All of the definitions and proofs follow an inductive structure.
Plan

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Lambda terms

\begin{verbatim}
data Λ : Set where
  ν    : V → Λ
  · ·   : Λ → Λ → Λ
  λχ   : V → Λ → Λ
\end{verbatim}

- One set of names for both bound and free variables without identifying alpha-equivalent terms.
Multiple Substitution

$$\Sigma = V \rightarrow \Lambda$$

- Functions mapping every variable to a term.
- Constructed from the identity substitution $$\iota : \Sigma$$ and an update operator $$\llt : \Sigma \rightarrow V \times \Lambda \rightarrow \Sigma$$
- The application of a substitution $$\sigma$$ to a term $$M$$ is noted as $$M \bullet \sigma$$ and defined by structural recursion on $$M$$.
- The case for the abstraction renames the abstraction variable according to $$\chi$$ which guarantees certain choice axioms:
  $$(\lambda x. M) \bullet \sigma = \lambda y. (M \bullet \sigma \llt (x, y))$$, where $$y = \chi(\sigma, \lambda x. M)$$, is the first variable not free in $$\sigma \downarrow M$$.
Alpha Conversion

```
data ~_α_ : Λ → Λ → Set where
  ~ν : {x : V} → (ν x) ~_α_ ν x
  ~· : {M M' N N' : Λ} → M ~_α_ M' → N ~_α_ N'
      → M · N ~_α_ M' · N'
  ~χ : {M M' : Λ}{x x' y : V}
      → y # χ x M → y # χ x' M'
      → M [ x := ν y ] ~_α_ M' [ x' := ν y ]
      → χ x M ~_α_ χ x' M'
```

- Alpha equivalent terms become equivalent when submitted to the same substitution.
data $\alpha$-star ($\rightsquigarrow : \Lambda \rightarrow \Lambda \rightarrow \text{Set} ) : \Lambda \rightarrow \Lambda \rightarrow \text{Set}$ where

refl : $\forall \{M\} \rightarrow \alpha$-star $\rightsquigarrow M M$

$\alpha$-step : $\forall \{M \ N \ N'\} \rightarrow \alpha$-star $\rightsquigarrow M \ N' \rightarrow N' \sim\alpha \ N \rightarrow \alpha$-star $\rightsquigarrow M \ N$

append : $\forall \{M \ N \ K\} \rightarrow \alpha$-star $\rightsquigarrow M \ K \rightarrow \rightsquigarrow K \ N \rightarrow \alpha$-star $\rightsquigarrow M \ N$

- One-step and transitivity can be proven from the previous definition.
Beta reducibility

\[
\begin{align*}
\text{data } & \_β@ : \Lambda \rightarrow \Lambda \rightarrow \mathbb{N} \rightarrow \text{Set} \text{ where} \\
& \text{outer-redex} : \forall \{x \ A \ B\} \rightarrow ((\lambda x \ A) \cdot B) \ β (A [x := B]) @ 0 \\
& \text{appNoAbsL} : \forall \{n \ A \ B \ C\} \rightarrow A β B @ n \rightarrow \neg \text{isAbs } A \\
& \quad \rightarrow (A \cdot C) β (B \cdot C) @ n \\
& \text{appAbsL} : \forall \{n \ A \ B \ C\} \rightarrow A β B @ n \rightarrow \text{isAbs } A \\
& \quad \rightarrow (A \cdot C) β (B \cdot C) @ (\text{suc } n) \\
& \text{appNoAbsR} : \forall \{n \ A \ B \ C\} \rightarrow A \ β \ B @ n \rightarrow \neg \text{isAbs } C \\
& \quad \rightarrow (C \cdot A) β (C \cdot B) @ (n + \text{countRedexes } C) \\
& \text{appAbsR} : \forall \{n \ A \ B \ C\} \rightarrow A β B @ n \rightarrow \text{isAbs } C \\
& \quad \rightarrow (C \cdot A) β (C \cdot B) @ (\text{suc } (n + \text{countRedexes } C)) \\
& \text{abs} : \forall \{n \ x \ A \ B\} \rightarrow A β B @ n \rightarrow (\lambda x \ A) β (\lambda x \ B) @ n \\
\end{align*}
\]

\[
\begin{align*}
& \_β : \Lambda \rightarrow \Lambda \rightarrow \text{Set} \\
& M β N = \sum x : \mathbb{N} (\lambda n \rightarrow M β N \suc n) \\
\end{align*}
\]

Equivalent to the classical inductive definition of beta reducibility.
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A sequence of $\beta$-reductions $A_0 \rightarrow_{n_1} A_1 \rightarrow_{n_2} \ldots \rightarrow_{n_k} A_k$ is called standard if $n_1 \leq n_2 \leq \ldots \leq n_k$.

We add an index to represent the lower bound of subsequent reductions, i.e. the number of the last redex reduced.

Allows performing explicit $\alpha$-conversion steps inside a reduction sequence.

**Theorem (Standardization)**

$$(\forall M, N) \ (M \rightarrow_\beta N \implies (\exists n) \ (seq_\beta M N n))$$
Head reduction in application

\[(\lambda x. A_0) A_1 A_2 \ldots A_n \rightarrow_{\text{hap}} A_0[x := A_1] A_2 \ldots A_n\]

| data _→hap_ : \(\Lambda \rightarrow \Lambda \rightarrow \text{Set} \) where
| hap-head : \(\forall\{x A B\} \rightarrow (\overline{x} \times A) \cdot B \rightarrow_{\text{hap}} (A [ x := B ])\)
| hap-chain : \(\forall\{C A B\} \rightarrow A \rightarrow_{\text{hap}} B \rightarrow (A \cdot C) \rightarrow_{\text{hap}} (B \cdot C)\)

| _→hap_ : \(\Lambda \rightarrow \Lambda \rightarrow \text{Set} \)
| _→hap_ = \(\alpha\text{-star} \_\rightarrow_{\text{hap}}\)

**Lemma**

\[(\forall M, N, \sigma) (M \rightarrow_{\text{hap}} N \implies M \bullet \sigma \rightarrow_{\text{hap}} N \bullet \sigma)\]
Kashima defines an inductive relation that captures the existence of a Standard Reduction Sequence between two terms.

```haskell
data _→→_ (L : Λ) : Λ → Set where
  st-var : ∀{x} → L → hap (v x) → L → st (v x)
  st-app : ∀{A B C D} → L → hap (A ∙ B) → A → st C → B → st D → L → st (C ∙ D)
  st-abs : ∀{x A B} → L → hap (x x A) → A → st B → L → st (x x B)
  st-α : ∀{K B} → L → st K → K ∼α B → L → st B
```

We now prove that:

\[ M \xrightarrow{\beta} N \implies M \xrightarrow{\text{st}} N \implies (\exists n) \ (\text{seq}_{\beta \text{st}} M N n) \]
Standard compatibility with substitution

Lemma

\[(\forall M, N, \sigma, \sigma') \ (M \rightarrow_{st} N \land \sigma \rightarrow_{st} \sigma' \implies M \bullet \sigma \rightarrow_{st} N \bullet \sigma')\]

- By induction on \(M \rightarrow_{st} N\)
- The case for the abstraction requires the use of multiple substitution in order to use the induction hypothesis.
Beta $\Rightarrow$ Standard

- $(\forall x, M, A, B) \ (M \rightarrow_{st} (\lambda x A) B \Rightarrow M \rightarrow_{st} A[x := B])$
- $(\forall M, N) \ (M \rightarrow_{st} N \land N \rightarrow_{\beta} P \Rightarrow M \rightarrow_{st} P)$

**Lemma**

$(\forall M, N) \ (M \rightarrow_{\beta} N \Rightarrow M \rightarrow_{st} N)$
Standard \implies Standard Sequence

- $(\forall M, N) \ (M \to_{hap} N \implies \text{seq}_{\beta st} M N 0)$
- $(\forall M, N, n, x) \ (\text{seq}_{\beta st} M N n \implies \text{seq}_{\beta st} (\lambda xM) (\lambda xN) n)$

**Lemma**

$(\forall M, N) \ (M \to_{st} N \implies (\exists n) \ (\text{seq}_{\beta st} M N n))$

Notice that the converse holds as well.
Theorem (Standardization)

\((\forall M, N) \ (M \to^\beta N \iff (\exists n) \ (\text{seq}_{\beta}st \ M \ N \ n))\)

- Follows directly from the previous lemmas.
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Leftmost Reduction Theorem
As a corollary of the Standardization Theorem

Theorem

If $M$ has a normal form, then the leftmost-outermost reduction strategy always finds it.

- Interesting metatheoretical result about reduction strategies.
- Beta-equality is decidable for normalizing terms.
Leftmost Reduction Theorem
Formalization in Agda

\[ \lambda \rightarrow l_\rightarrow : \Lambda \rightarrow \Lambda \rightarrow \text{Set} \]
\[ M \lambda \rightarrow l N = M \beta N \downarrow 0 \]

\[ \lambda \rightarrow l_\rightarrow : \Lambda \rightarrow \Lambda \rightarrow \text{Set} \]
\[ \lambda \rightarrow l_\rightarrow = \alpha \text{-star } \lambda \rightarrow l_\rightarrow \]

\[ \text{nf} : \Lambda \rightarrow \text{Set} \]
\[ \text{nf} M = \text{countRedexes} M \equiv 0 \]

**Theorem**

\[(\forall M, N) (M \rightarrow \beta N \land \text{nf} N \implies M \rightarrow l N)\]
Leftmost Reduction Theorem

Proof

Lemma

\((\forall M, N, n) \ (M \ \beta \ N \ @ \ n \ \land \ \text{nf } N \ \Longrightarrow \ n \equiv 0)\)

- By induction on \(M \ \beta \ N \ @ \ n\)

Lemma

\((\forall M, N, n) \ (\text{seq}^{\beta, \text{st}} M N n \ \land \ \text{nf } N \ \Longrightarrow \ M \ \rightarrow_1 N)\)

- By induction on \(\text{seq}^{\beta, \text{st}} M N n\) using the previous lemma for the case \(\beta - \text{step}\).

- Now the Leftmost Reduction Theorem follows directly from \(M \ \rightarrow_\beta N \ \Longrightarrow \ (\exists n) \ (\text{seq}^{\beta, \text{st}} M N n) \ \Longrightarrow \ M \ \rightarrow_1 N\), for \(N\) in normal form.
Conclusions

- Kashima’s proof is correct! (completely certified in Agda).
- Using Stoughton’s substitution, the theorem only requires structural induction. Novel in relation to previous approaches:
  - McKinna and Pollack (1999)
  - Guidi (2012)
  - Emerich and Ignas Vysniauskas (2014)
- Only a few lemmas had to be added to the substitution library in order to prove the theorem.
- Proof of equivalence between Kashima’s notion of beta-reducibility and the classical one.
- Introduction of a new inductive definition of a standard reduction sequence, namely $\text{seq}_\beta^\text{st}$.
- Leftmost Reduction Theorem
Thank you!